Dual Extremum Principles in Finite Deformation Theory with Applications to Post-Buckling Analysis of Extended Nonlinear Beam Model

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The critical points of the generalized complementary energy variational principles are clarified. An open problem left by Hellinger and Reissner is solved completely. A pure complementary energy (involving the Kirchhoff-type stress only) is constructed. We prove that the well-known generalized Hellinger-Reissner's energy $L(u,s)$ is a saddle point functional if and only is the Gao-Strang gap function is positive. In this case, the system is stable and the minimum potential energy principle is equivalent to a unique maximum dual variational principle. However, if this gap function is negative, then $L(u,s)$ is a so-called $\partial^c$-critical point functional. In this case, the system has two extremum complementary principles. An interesting triality theorem for nonconvex variational problem is discovered, which can be used to study nonlinear bifurcation problems, phase transitions, variational inequality, and other things.

In order to study the shear effects in frictional post-buckling problems, a new second order 2-D nonlinear beam model is developed. Its total potential is a double-well energy. A stability criterion for post-buckling analysis is proposed, which shows that the minimax complementary principle controls a stable buckling state. The unilateral buckling state is controlled by a minimum complementary principle. However, the maximum complementary principle controls the phase transition.

INTRODUCTION

The complementary energy principle was first proposed by Hellinger in 1914. Since the boundary conditions were clarified by Reissner, the dual variational principles and methods in finite deformation mechanics have been studied extensively by many mechanicians (cf. e.g. Koiter, 1976; Nemat-Nasser, 1977; Atluri, 1980; Lee & Shield, 1980; Boller, 1983; Tabaark, 1984; Ogden & Reddy, 1983; Ogden, 1984 and much more). It is known that the Hellinger-Reissner principle involves both the Kirchhoff stress and the displacement, so it is not considered as a pure complementary energy principle. For more than 80 years, this principle has been only realized as a stationary principle. The extremum property of this principle has been an open problem, which yielded many arguments. The Levinson & Zubov principle involves only the Piola stress, but this principle is not valid unless the constitutive relation is invertible. Unfortunately, in finite deformation theory, the stored energy is usually nonconvex in the deformation gradient. The convexity conditions and related topics in potential energy variational problems have been discussed by many mathematicians (cf. e.g. Dacorogna, 1989). The Fraeijs de Veubeke Principle is always true. But it involves both the Piola stress and the rotation tensor. Moreover, the Piola tensor is not symmetric. It is very difficult to use these principles.

Duality theory in geometrical linear systems has been well studied for both convex and nonconvex variational problems (cf. e.g. Rockafellar, 1974; Ekeland & Temam, 1976; Aucmity, 1983). The symmetry between the primal and dual energy principles is amazing beautiful (see Strang, 1986; Sewell, 1987; Marsden & Ratiu, 1995). However, in geometrical nonlinear systems, such a symmetry is lost because of nonlinearity of the finite deformation operator.

In order to recover this broken symmetry in finite deformation theory, a so-called complementary gap function was discovered by Gao & Strang in 1989, and a general duality was established in geometrical nonlinear systems. They proved that if this gap function is positive on the statically admissible field, the generalized complementary energy is a saddle point functional, the total potential is convex, and its dual problem is concave. Applications of this general duality theory have been given in a series of publications on finite elastoplasticity (see Gao et al, 1989-95). Some open problems in large deformation plastic limit analysis were solved (Gao & Strang, 1989b, Gao, 1994). While the duality theory for unstable systems (where the gap function is non-positive) has remained open.

The purpose of this paper is to solve the open problems left by Hellinger-Reissner. The original motivation of this research was the desire to complete Gao & Strang work on the duality theory in geometrical nonlinear systems. In the recent research on the post-buckling analysis (Gao-
Russell, 1996), it was realized that this research is directly related to those phase transitions, smart materials, variational inequality, nonlinear bifurcation and stability analysis, and others. The total potential in these systems is usually a multi-well energy. The direct approaches and relaxation methods have been discussed extensively in recent years (cf. e.g. Panagiotopoulos, 1985; Ball-James, 1987; Kohn, 1991; Fried-Gurtin, 1996 and Steinmann, 1996, and others).

In this paper, a complete duality theory for this non-convex variational problem is established, which includes three extremum complementary principles and an interesting intrinsic tracity theorem. The critical point of the generalized variational principle is clarified.

It is interesting to note that the well-known von Karman model in 1-D case is actually linear (Gao, 1996), and can be used only for pre-buckling analysis. In order to study the frictional unilateral post-buckling problem, a new 2D extended beam model is proposed, where the shear deformation is allowed to vary in the lateral direction. The total potential energy is a double-well functional. Application of the theory proposed in this paper is illustrated by the post-buckling analysis of this beam model. Moreover, a general analytic solution for nonlinear equilibrium problem of Ericksen’s bar is obtained in the last section.

**BASIC EQUATIONS AND GAP FUNCTION**

Let \( \Omega \subset \mathbb{R}^3 \) be an open, simply connected, bounded domain with boundary \( \partial \Omega = \Gamma_i \cup \Gamma_u, \Gamma_i \cap \Gamma_u = \emptyset \). On \( \Gamma_i \), the surface traction \( \hat{t} \) is given; while on the remaining part \( \Gamma_u \), the displacement is prescribed. Let \( \mathcal{U} \) be a general displacement space, its conjugate space \( \mathcal{F} \) should be a general force space. The bilinear form \( (\cdot , \cdot) : \mathcal{F} \times \mathcal{U} \to \mathbb{R} \) puts the spaces \( \mathcal{U} \) and \( \mathcal{F} \) in duality. If \( f \in \mathcal{F} \) is specified as the body force \( \mathbf{B} \) in \( \Omega \), and the surface traction \( \hat{t} \) on \( \Gamma \), then this bilinear form

\[
(f, u) = \int_\Omega u \cdot \mathbf{B} \, d\Omega + \int_\Gamma u \cdot \hat{t} \, d\Gamma
\]

represents the external work. The general strain space and its conjugate stress space are denoted by \( \mathcal{E} \) and \( \mathcal{S} \), respectively. The spaces \( \mathcal{E} \) and \( \mathcal{S} \) are put in duality by another bilinear form \( (\cdot , \cdot) : \mathcal{S} \times \mathcal{E} \to \mathbb{R} \):

\[
(s, e) = \int_\Omega s \cdot e \, d\Omega = \int_{\Gamma} e_{ij} s_{ij} \, d\Omega,
\]

which represents the internal work.

By introducing a convex, differentiable stored strain energy function: \( W : \mathcal{E} \to \mathbb{R} \), the constitutive equation can be described as

\[
s = Dw(e),
\]

where \( Dw(e) = \partial W/\partial e \) stands for the Gâteaux-derivative of \( W \). The complementary energy can be obtained uniquely by the Legendre-Fenchel transformation:

\[
W^*(s) := \sup_{e} \{ s : e - W(e) \},
\]

which is always convex. Since \( W \) is convex, we have:

\[
s = \frac{\partial W}{\partial e} \Leftrightarrow e = \frac{\partial W}{\partial s} \Leftrightarrow W(e) + W^*(s) = s : e
\]

In the finite deformation theory, the general strain \( e \) can be described by the abstract geometrical equation:

\[
e = \Lambda u,
\]

where \( \Lambda : \mathcal{U} \to \mathcal{E} \) is a finite deformation operator. For any given \( e = \Lambda u \in \mathcal{E} \), the directional derivative of \( e \) at \( \hat{u} \) in the direction \( u \in \mathcal{U} \) is defined as

\[
\delta e(\hat{u}, u) := \lim_{\theta \to 0^+} \frac{e(\hat{u} + \theta u) - e(\hat{u})}{\theta} = \Lambda_{f}(\hat{u}) u,
\]

where \( \Lambda_{f} \) is the Gâteaux derivative of the operator \( \Lambda \) at \( \hat{u} \). According to (Gao & Strang, 1989a), we have the following decomposition:

\[
\Lambda = \Lambda_{f} + \Lambda_{n},
\]

where \( \Lambda_{n} \) is the complementary operator of \( \Lambda_{f} \), which plays a central role in finite deformation theory.

The relation between the bilinear forms \( (\cdot , \cdot) \) and \( (\cdot , \cdot)^* \) can be given as

\[
(s, \Lambda u) = (\Lambda^*_f s, u) - G(u, s),
\]

where \( \Lambda^*_f : \mathcal{S} \to \mathcal{F} \) is the adjoint operator of \( \Lambda_f \), defined by the generalized Gauss-Green theorem:

\[
(s, \Lambda_f(u)v) = (\Lambda_f^*(u)s, v),
\]

where \( G \) is the so-called complementary gap function introduced by Gao & Strang in 1989:

\[
G(u, s) = (s, -\Lambda_n u).
\]

The relation between the bilinear forms \( (\cdot , \cdot) \) and \( (\cdot , \cdot)^* \) can be given as

\[
(s, \delta e(\hat{u}, u)) = (s, \Lambda_{f}(\hat{u}) u) = (\bar{f}, u) \quad \forall u \in \mathcal{U},
\]

the equilibrium equation can be obtained as:

\[
\Lambda_{f}^*(u)s = \bar{f}.
\]

Let \( \mathcal{U}_{a} \subset \mathcal{U} \) be a kinematically admissible space, then the general mixed boundary value problem ((BVP) for short) can be proposed as below:

**Problem 1 (BVP)** For the given external force \( \bar{f} = \{ \hat{t}(\text{in } \Omega), \hat{t}(\text{on } \Gamma_i) \} \), find \( u \in \mathcal{U}_{a} \) such that

\[
\Lambda_{f}^* DW(\Lambda u) = \bar{f}.
\]

As an example, for the Green’s strain tensor \( e = E \):

\[
E = \Lambda u = \frac{1}{2} [\nabla u + (\nabla u)^t] + \frac{1}{2} [\nabla \hat{u}]^t [\nabla \hat{u}]^t,
\]

\( \Lambda \) is a quadratic operator, and

\[
\Lambda_{f}(\hat{u}) u = \frac{1}{2} [\nabla u + (\nabla u)^t] + \frac{1}{2} [\nabla \hat{u}]^t [\nabla \hat{u}] [\nabla \hat{u}]^t [\nabla \hat{u}],
\]

where \( \hat{u} \) is a general force space. The bilinear form \( (\cdot , \cdot) \) and \( (\cdot , \cdot)^* \) can be given as

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\( \Lambda \) is a quadratic operator, and

\[
\Lambda_{f}(\hat{u}) u = \frac{1}{2} [\nabla u + (\nabla u)^t] + \frac{1}{2} [\nabla \hat{u}]^t [\nabla \hat{u}] [\nabla \hat{u}]^t [\nabla \hat{u}],
\]
\[ \Lambda_0 (\tilde{u}) u = - \frac{1}{4} [ (\nabla \tilde{u})^T (\nabla u) + (\nabla u)^T (\nabla \tilde{u}) ] .\]

For hyperelasticity, the strain energy \( W \) should be a convex function of \( \mathbf{E} \), and the stress \( \mathbf{s} \) conjugate to \( \mathbf{E} \) is the Kirchhoff stress tensor \( \mathbf{S} = \partial W / \partial \mathbf{E} \). The gap function is then given by

\[ G(u, S) = \int_{\Omega} \frac{1}{2} \text{tr}[(\nabla u)^T S(\nabla u)] \, d\Omega . \tag{12} \]

The equilibrium operator \( \Lambda_t^* \) should be

\[ \Lambda_t^* (u) S = \begin{cases} - \nabla : [ (I + \nabla u) S ] & \text{in } \Omega, \\ n : [ (I + \nabla u) S ] & \text{on } \Gamma, \end{cases} \tag{13} \]

where \( \mathbf{n} \) is the unit vector normal to the boundary \( \Gamma \). The gap function for the Seth-Hill strain tensors and the deformation gradient were discussed by Gao (1992).

According to convex analysis, \( W^* \) is uniquely determined by the Legendre-Fenchel transformation (2). But in the paper by Teleaga (1980), instead of (6), he expressed the Green strain tensor using two independent variables \( \mathbf{p}, \mathbf{q} \), (i.e. \( \mathbf{E} = \mathbf{p} + \frac{1}{2} \mathbf{q}^T \mathbf{q} \)) where there is really only one: (11) with \( \mathbf{p} = (\mathbf{q} + \mathbf{q}^T ) / 2 \), \( \mathbf{q} = \nabla \mathbf{u} \). Also, introducing a new conjugate tensor \( \mathbf{T} \) of \( \mathbf{q} \) (with no apparent justification), this leads to his formula (i.e. equation (22) in his paper):

\[ g^* (S, T) = \sup_{\mathbf{p}, \mathbf{q}} \{ \mathbf{S} : \mathbf{p} + \mathbf{T} : \mathbf{q} - W(\mathbf{p} + \frac{1}{2} \mathbf{q}^T \mathbf{q}) \} = W^*(S) + g_1^*(S, T), \]

which contains a spurious term \( g_1^* = \sup_{\mathbf{q}} \{ \mathbf{T} : \mathbf{q} - \frac{1}{2} \mathbf{S} : (\mathbf{q}^T \mathbf{q}) \} \). This false term leads to other erroneous conclusions.

**POTENTIAL ENERGY PRINCIPLE**

In the following analysis, we assume that \( \Lambda \) is a quadratic operator. Let \( F(u) \) be the external work:

\[ F(u) = \int_{\Omega} \tilde{b} : u \, d\Omega + \int_{\Gamma} \tilde{c} : u \, d\Gamma . \tag{14} \]

The total potential functional \( P : \mathcal{U}_a \to \mathbb{R} \) is defined by

\[ P(u) := \int_{\Omega} W(\Lambda u) \, d\Omega - F(u) . \tag{15} \]

So the minimal potential problem (inf primal problem) associated with (BVP) is to find \( \tilde{u} \) such that

\[ (P_{inf}) : \quad P(\tilde{u}) = \inf_P (u) \quad \forall u \in \mathcal{U}_a . \tag{16} \]

It is easy to prove that the stationary condition \( \delta P(\tilde{u}; u) = 0 \quad \forall u \in \mathcal{U}_a \) is equivalent to (10). But the stationary point is not identical to the minimizer of \( P \) because \( P \) may not be convex. According to Gao & Strang (1989a), one has the following extremum principle:

**Theorem 1** For any given critical point \( \tilde{u} \) of \( P \) and associated general stress tensor \( \mathbf{s} = D W(\Lambda \tilde{u}) \), if

\[ G(u, \mathbf{s}(\tilde{u})) \geq 0 \quad \forall u \in \mathcal{U}_a , \tag{17} \]

then \( \tilde{u} \) minimizes \( P \) over \( \mathcal{U}_a \). If \( \mathcal{U}_a \) is a closed, convex subset of a reflexive Banach space, then the solution of the problem \( (P_{inf}) \) exists. If the gap function is strictly positive, the solution is unique.

This theorem shows that if the gap function has a positive sign, the system is stable. However, if the gap function has a negative sign, the total potential energy is nonconvex. Then \( P \) may have a local maximizer on a subset of \( \mathcal{U} \), which could be a critical point in phase transitions. Let \( \mathcal{U}_b \) be a subset of \( \mathcal{U}_a \), we can propose the sup-primal problem: to find \( \tilde{u} \in \mathcal{U}_b \) such that

\[ (P_{sup}) : \quad P(\tilde{u}) = \sup_{\mathcal{U}_b} P(u) \quad \forall u \in \mathcal{U}_b . \tag{18} \]

According to Gao & Strang (1989a), the total complementary energy should be:

\[ P^c (s, u) := F^*(\Lambda_t^* (u) s) - \int_{\Omega} W^*(s) \, d\Omega - G(u, s) , \tag{19} \]

where \( F^* \) is the conjugate function of \( F \). For example, if \( \mathcal{U}_a = \{ u \in \mathcal{U} \mid u = 0 \text{ on } \Gamma \} \):

\[ F^*(f) = \inf_{u \in \mathcal{U}_a} \{ (f, u) - F(u) \} = \begin{cases} 0 & \text{if } f = \tilde{f} \\ -\infty & \text{if } f \neq \tilde{f} \end{cases} \]

Let \( S_a \subset S \) be a range of \( DW(\mathbf{c}) \). We introduce the so-called equilibrium admissible space:

\[ S_0 := \{ (u, s) \in \mathcal{U}_a \times S_0 \mid \Lambda_t^* (u) s = \tilde{f} \} , \tag{20} \]

then on \( S_0, P^c \) is finite. The stationary condition

\[ \delta P^c (\tilde{s}, \tilde{u}; s, u) = 0 \quad \forall (s, u) \in S_0 \]

is also equivalent to (10). We have the stationary complementary energy principle:

**On \( S_0, the stationary point \( (\tilde{s}, \tilde{u}) \) of \( P^c \) solves (BVP).**

If \( \Lambda u = \mathbf{F} \) is the Green strain, \( P^c (s, u) \) is the well-known Hellinger-Reissner complementary energy. But its two variables \( s \) and \( u \) are not independent. To relax the equilibrium constraint, the Lagrangian has to be introduced. If \( \Lambda u = \mathbf{F} \), the deformation gradient, then \( P^c \) is the Levinson-Zubov energy, but its stationary condition is not equivalent to (BVP), since the equivalent relation (3) is not true for \( \mathbf{e} = \mathbf{F} \), which is not a strain measure.

**GENERALIZED COMPLEMENTARY VARIATIONAL PRINCIPLES**

Using the Legendre transformation \( F^* (\Lambda_t^* (u) s) = (\Lambda_t^* (u) s, u) - F(u) \) to replace \( F^* \) in \( P^c \), the Lagrangian \( L : \mathcal{U} \times S \to \mathbb{R} \) can be given as

\[ L(u, s) := (\Lambda u, s) - \int_{\Omega} W^*(s) \, d\Omega - F(u) . \tag{21} \]
This is the so-called generalized complementarity energy. A point \((\hat{u}, \hat{s}) \in \mathcal{U} \times \mathcal{S}\) is said to be a critical point of \(L\) if
\[
D_u L(\hat{u}, \hat{s}) = 0, \quad D_s L(\hat{u}, \hat{s}) = 0.
\]
Here \(D_u, D_s\) denote partial Gâteaux-derivatives on \(\mathcal{U}\) and \(\mathcal{S}\), respectively. Then it is easy to find that
\[
D_u L(\hat{u}, \hat{s}) = 0 \implies \Lambda^*_u(\hat{u}) - DF(\hat{u}) = 0, \quad (22)
\]
\[
D_s L(\hat{u}, \hat{s}) = 0 \implies \Lambda^* - DW^*(\hat{s}) = 0. \quad (23)
\]
So we have the generalized variational principle:

For any given \((u, s) \in \mathcal{U}_a \times \mathcal{S}_a\), the critical point \((\hat{u}, \hat{s})\) solves the (BVP).

In geometrical linear systems, the critical points of the Lagrangian was clarified by Aucumuty (1983). For geometrical nonlinear systems, we need following definitions:

**Definition 1** A point \((\hat{u}, \hat{s})\) is said to be a saddle point of \(L\) if
\[
L(\hat{u}, \hat{s}) \leq L(u, s) \leq L(\hat{u}, \hat{s}) \quad \forall (u, s) \in \mathcal{U} \times \mathcal{S}. \quad (24)
\]
A point \((\hat{u}, \hat{s})\) is said to be a \(\partial^-\)-critical point of \(L\) if
\[
L(\hat{u}, \hat{s}) \geq L(u, s) \leq L(\hat{u}, \hat{s}) \quad \forall (u, s) \in \mathcal{U} \times \mathcal{S}. \quad (25)
\]
A point \((\hat{u}, \hat{s})\) is said to be a \(\partial^+\)-critical point of \(L\) if
\[
L(\hat{u}, \hat{s}) \leq L(u, s) \geq L(\hat{u}, \hat{s}) \quad \forall (u, s) \in \mathcal{U} \times \mathcal{S}. \quad (26)
\]

**Remark** According to convex analysis, the inequality (25) is equivalent to the partial sub-differential of \(L\):
\[
0 \in \partial_u L(\hat{u}, \hat{s}), \quad 0 \in \partial_s L(\hat{u}, \hat{s}),
\]
which is the Aucumuty’s definition of \(\partial\)-critical point in geometrical linear system (Aucumuty, 1983). While the inequality (26) is equivalent to the partial super-differential of \(L\):
\[
0 \in \partial^+_u L(\hat{u}, \hat{s}), \quad 0 \in \partial^+_s L(\hat{u}, \hat{s}).
\]
If \(\Lambda\) is linear, we can define a Hamiltonian \(\hat{H} = (\Lambda u, s) - L(u, s)\) such that the canonical form \(\Lambda \hat{u} \in \lambda H(\hat{u}, \hat{s}), \Lambda^* \hat{s} \in \partial_s H(\hat{u}, \hat{s})\) holds, which is Aucumuty’s definition of anomalous critical point. Unfortunately this symmetrical canonical form does not hold in geometrical nonlinear systems.

**Theorem 2** Suppose that \((\hat{u}, \hat{s})\) is a critical point of \(L\). Then \((\hat{u}, \hat{s})\) is a saddle point of \(L\) if and only if
\[
G(u, s) \geq 0 \quad \forall u \in \mathcal{U}_a; \quad (\hat{u}, \hat{s}) \text{ is a } \partial^+\text{-critical point of } L\text{ if and only if } G(u, s) \leq 0 \quad \forall u \in \mathcal{U}_a.
\]

**Proof** Since \(L(u, s) : S \to \mathbb{R}\) is concave for any given \(u \in \mathcal{U}\), if \((\hat{u}, \hat{s})\) is a critical point of \(L\), we have
\[
L(\hat{u}, \hat{s}) \leq L(u, s) \quad \forall s \in \mathcal{S}. \quad (27)
\]
Since \(\Lambda\) is quadratic, for any given \(u = \tilde{u} + \delta u\),
\[
\Lambda(u) u = \Lambda(\tilde{u}) \tilde{u} + \Lambda(\tilde{u}) \delta u - \Lambda(\delta u) \delta u.
\]
Because \(F\) is linear, \(F(\tilde{u} + \delta u) = F(\tilde{u}) + \langle DF(\tilde{u}), \delta u \rangle\).

Then
\[
L(u, s) - L(\hat{u}, \hat{s}) = \langle \Lambda^*_u(\hat{u}) - DF(\hat{u}), \delta u \rangle + G(\delta u, \hat{s})
\]
\[
= \langle D_u L(\hat{u}, \hat{s}), \delta u \rangle + G(\delta u, \hat{s}).
\]
Proof. Suppose that \((\tilde{u}, \tilde{s})\) is a saddle point of \(L\). Since \(L : S \rightarrow \mathbb{R}\) is concave,
\[
P(u) = \sup_s L(u, s) = L(u, \tilde{s}) \geq L(\tilde{u}, \tilde{s}) \quad \forall u \in U_u.
\]
By Theorem 3, \(P(u) \geq P(\tilde{u})\) \(\forall u \in U_u\). So \(\tilde{u}\) is a solution of \((P_{inf})\). Since the gap function is positive on \(S_u\), \(L : U_u \rightarrow \mathbb{R}\) is convex, then by Lemma 1,
\[
P^*(s) = \inf_u L(u, s) = L(\tilde{u}, s) \leq L(\tilde{u}, \tilde{s}) = P^*(\tilde{s}) \quad \forall s \in S_a.
\]
This shows that \(\tilde{s}\) is a solution of \((P_{sup})^*\) and the Theorem 3 gives that \(P(\tilde{u}) = P^*(\tilde{s})\).

Conversely, if \(\tilde{u}\) is a solution of \((P_{inf})\), \(\tilde{s}\) is a solution of \((P_{sup})^*\) and \(P(\tilde{u}) = P^*(\tilde{s})\), we have, from the definition of \(L\) and \(P^*\),
\[
\sup_s L(\tilde{u}, s) = \inf_u L(u, \tilde{s}) \leq L(\tilde{u}, \tilde{s}) = P^*(\tilde{s}) \quad \forall s \in S_a.
\]
Thus
\[
L(\tilde{u}, \tilde{s}) \leq L(\tilde{u}, s) \quad \forall s \in S_a.
\]
Similarly,
\[
\inf_u L(u, \tilde{s}) = \sup_s L(\tilde{u}, s) \geq L(\tilde{u}, \tilde{s}) = P^*(\tilde{s}) \quad \forall s \in S_a.
\]
So
\[
L(u, s) \geq L(\tilde{u}, \tilde{s}) \quad \forall u \in U_u.
\]
Thus \((\tilde{u}, \tilde{s})\) is a saddle point of \(L\).

This theorem shows that for a given \(\Lambda\), if the gap function is positive on \(S_u\), the system has only one potential extremum principle \((P_{inf})\) and only one complementary extremum principle\(^1\) \((P_{sup})^*\). However, if the gap function is negative on \(S_u\), the system may have more than one primal-dual problems. Let \(U_b, S_b\) be the subspaces of \(U_d\) and \(S_d\), respectively, such that \(U_b \times S_b \subset S_u\). Then for nonconvex systems, we have the following results.

**Theorem 5 (Maximum Complementary Principle)**
Suppose that the gap function \(G(u, s) \leq 0 \quad \forall (u, s) \in U_b \times S_b \subset S_u\), then on \(U_b \times S_b\), \(\sup_{u \in U_b} P(u) = \sup_{s \in S_b} P^*(s)\).

A point \((\bar{u}, \bar{s})\) maximizes \(L\) on \(U_b \times S_b\) if and only if
\[
P(\bar{u}) = \sup_{u \in U_b} P(u) \quad \text{and} \quad P^*(\bar{s}) = \sup_{s \in S_b} P^*(s). \tag{36}
\]

**Theorem 6 (Minimum Complementary Principle)**
Suppose that \((\bar{u}, \bar{s})\) is a critical point of \(L\), and \(U_b \times S_b \subset S_u\) is a neighborhood such that \(P\) and \(P^*\) have only one critical point on \(U_b\) and \(S_b\), respectively. If the gap function \(G(\bar{u}, \bar{s}) \leq 0\), then \(\bar{u}\) minimizes \(P\) on \(U_b\) if and only if \(\bar{s}\) minimizes \(P^*\) on \(S_b\).

In nonconvex systems, the Lagrangian could have several critical points. The gap function could be positive on one critical point and be negative on other. Combining the above results, we have following interesting result:

\(^1\) The multi-duality for different \(\Lambda\) was discussed by Gao & Yang, 1995.

**Theorem 7 (Triality Theorem)** Suppose that \((\bar{u}, \bar{s})\) is a critical point of \(L\), and \(U_b \times S_b \subset S_u\) is a neighborhood such that \(P\) and \(P^*\) have only one critical point on \(U_b\) and \(S_b\), respectively. If \(G(\bar{u}, \bar{s}) = 0\), then
\[
P(\bar{u}) = \inf_{u \in U_b} P(u) \quad \Leftrightarrow \quad P^*(\bar{s}) = \sup_{s \in S_b} P^*(s); \tag{37}
\]
If \(G(\bar{u}, \bar{s}) \leq 0\), then either
\[
P(\bar{u}) = \inf_{u \in U_b} P(u) \quad \Leftrightarrow \quad P^*(\bar{s}) = \inf_{s \in S_b} P^*(s), \tag{38}
\]
or
\[
P(\bar{u}) = \sup_{u \in U_b} P(u) \quad \Leftrightarrow \quad P^*(\bar{s}) = \sup_{s \in S_b} P^*(s). \tag{39}
\]

The proofs of these theorems are given elsewhere. From the triality theorem, it is easy to get the following result:

**Corollary 2**
If \(G(u, s) \geq 0 \quad \forall (u, s) \in S_u\), then
\[
\inf_{u \in U_b} \sup_{s \in S_b} L(u, s) = \sup_{u \in U_b} \inf_{s \in S_b} L(u, s). \tag{40}
\]
If \(G(u, s) \leq 0 \quad \forall (u, s) \in S_u\) then
\[
\inf_{u \in U_b} \sup_{s \in S_b} L(u, s) = \sup_{u \in U_b} \inf_{s \in S_b} L(u, s). \tag{41}
\]

The equality
\[
\sup_{u \in U_b} \inf_{s \in S_b} L(u, s) = \inf_{u \in U_b} \sup_{s \in S_b} L(u, s)
\]
is trivial on \(U \times S\). But on a neighborhood of the critical points, it gives the maximum complementary principle (39). In the Lagrangian \(L\), if we replace \(\bar{W}^*\) by the Legendre transformation (2), then we have the so-called pseudo-Lagrangian (see Gao & Strang, 1989a):
\[
L(\bar{u}, s, e) = \langle \lambda u - e, s \rangle + \int_{\Omega} W(e) \, d\Omega - F(u). \tag{42}
\]

Obviously we have
\[
L(u, s) = \inf_{e} L_p(u, s, e). \tag{43}
\]

If \(\Lambda\) is the Green strain operator, this is the well-known Hu-Washizu generalized potential energy. Its extremum property follows easily from Corollary 2.

**NONLINEAR BEAM THEORY**

As long ago as 1837 Saint-Venant had realized that beam cross sections do not remain plane in bending. In order to study the influence of shear deformation on large deformed thick beam solutions, the following deformation model was introduced in Gao & Russell, (1996) in 2D domain \(\Omega = \{(x, y) \in [0, L] \times [-h, h]\}:
\[
\xi(x, y) = u_0(x) - y\theta(x) + v(x, y), \quad \eta(x) = w(x),
\]
where \(u_0(x)\) is the horizontal displacement of the middle axis \(y = 0\), \(\theta\) is the bending angle, \(v(x, y)\) is the
shear deformation. Based on this deformation model, a nonlinear theory for extended Timoshenko beam was proposed:

\[ Iw_{xxxxx} - (aw_{x}^4 - \lambda)w_{xx} = f(x) + \int_{-h}^{h} yv_{xxx}dy, \]

where \( a > 0, \beta = (1 - \nu)/2 \) are material constants, \( I \) is the moment of inertia per unit length, \( \lambda = (1 - \nu^2)p/E, \) \( p \) is the axial compress load, and \( w_{xx}^2 = (w_x)^2. \) If the shear deformation can be ignored, then we should have

\[ Iw_{xxxxx} - \alpha w_{x}^2 w_{xx} + \lambda w_{xx} = f. \]

This nonlinear beam model was obtained in (Gao, 1996) by considering the stress in the lateral direction. In the present paper, instead of shear deformation \( v(x, y) \), we use directly the axial displacement \( u(x, y) \) as the unknown function. The displacement vector is then \( u(x, y) = (u(x, y), w(x, y)) \) and the Green strain is

\[ E = \begin{bmatrix} \frac{1}{2}(u_{xx}^2 + w_{xx}^2) & \frac{1}{2}(u_{yy} + w_{xx} + u_{xx}u_{yy}) \\ \frac{1}{2}(u_{yy} + w_{xx} + u_{xx}u_{yy}) & \frac{1}{2}w_{yy}^2 \\ w_{xx} + u_{yy} \end{bmatrix}. \]

For moderately large beam deflection problems, we may assume that \( h/L \sim w(x) \in O(1), u \sim \nu \sim w_{xx} \in O(\varepsilon), \) and \( w_{xx} \sim v_{x} \sim v_{y} \sim w_{xx} \in O(\varepsilon), \) where the notation \( \sim \) stands for “same order of magnitude”. By the Taylor expansion, we have \( \theta(x) = \tan^{-1}\frac{w_{xx}}{w_{yy}} = w_{xx} + O(\varepsilon^3). \)

Neglecting terms higher than \( O(\varepsilon^3) \) and using the engineering strain tensor notations: \( \varepsilon_x = \varepsilon_{xx}, \varepsilon_y = \varepsilon_{yy}, \gamma = 2\varepsilon_{xy}, \) we then have

\[ \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{bmatrix} = \begin{bmatrix} u_x + \frac{1}{2}w_{xx} \\ \frac{1}{2}w_{yy} \\ w_{xx} + u_{yy} \end{bmatrix}. \]

In the case of large displacements but small strain plane elastic deformations, the elastic constitutive relation can be given as

\[ \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma \end{bmatrix}. \]

Suppose the beam is subjected to a compressive axial load \( p \) at \( x = L, \) the distributed load \( f(x) = (q^+(x), f(x))^T \) on the top, and \( (q^-(x), 0)^T \) on the bottom of the beam. By the virtual work principle, the equilibrium equations and variational boundary conditions can be obtained as:

\[ \begin{align*}
\sigma_{x,x} + \tau_{y} &= 0, \quad \forall(x, y) \in \Omega, \\
\int_{-h}^{h} [(\sigma_{x} + \sigma_{y})w_{x} + \tau_{x}]dy + f(x) &= 0, \quad x \in [0, L], \\
\tau(x, h) = q^+(x) - q^-(x) &= 0, \quad x \in [0, L].
\end{align*} \]

(46)

The total potential energy of this problem is:

\[ P(v, w; \lambda) = \frac{1}{2} \int_{\Omega} [(v_{x} + w_{x}^2 - \lambda)^{2} + \beta(v_y + w_{xx})^2] d\Omega - \int_{0}^{L} \int_{0}^{h} \frac{1}{2} (\sigma_{x} + \sigma_{y})d\sigma_{x}dy. \]

(50)

It is surprising that if the shear effect is ignored, then

\[ P(w; \lambda) = \frac{1}{2} \int_{0}^{L} (\frac{1}{2} w_{x}^2 - \lambda)^{2} - \int_{0}^{L} f(x)w(x)dx. \]

(51)
is Ericksen's double-well energy in the extension of a
elastic bar (Ericksen, 1975). For a given vertical load
\( f \), it may have two minimizers for two possible buckling
states, respectively. Since \( P(v, w; \lambda) \) is a strictly convex
functional of \( v \), for any given \( w \), \( \inf_v P(v, w; \lambda) \) has
a unique solution \( \bar{v} \). But \( P(v, w; \lambda) \) is nonconvex in \( w \),
the solution for \( \inf_v P(v, w; \lambda) \) is not unique.

In order to establish a simple complementary energy
variational principle, we need to find the right geomet-/7/0 MECHANICS PAN-AMERICA 1997
rical mapping and dual variables. Instead of (44), we
let \( e = (e, \gamma) \) be the generalized strain vector, which
is given by the following geometrical equation:

\[
e = \begin{pmatrix} e \\ \gamma \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{1}{2} \frac{\partial w_x}{\partial y} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \Lambda(w)u.
\]

(52)

In this case,

\[
\Lambda(w) = \begin{pmatrix} \frac{\partial}{\partial y} & \frac{1}{2} \frac{\partial w_x}{\partial y} \\ \frac{1}{2} \frac{\partial w_x}{\partial y} & 0 \end{pmatrix}, \quad \Lambda_n(w) = \begin{pmatrix} 0 & -\frac{\partial}{\partial y} \\ \frac{1}{2} \frac{\partial w_x}{\partial y} & 0 \end{pmatrix}.
\]

Since \( \Lambda \) is quadratic, \( e \) should be a Green-type strain
vector. The strain energy \( W \) can be written as

\[
W(e; \lambda) = \frac{1}{2}(e - \lambda)^2 + \beta \gamma^2.
\]

The Kirchhoff-type stress vector \( s \) then is given by

\[
s = \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = DW(e; \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} e - \lambda \end{pmatrix}.
\]

(53)

Since \( W(e; \lambda) \) is a quadratic function of \( e \), by (2), its
conjugate function is uniquely defined as

\[
W^*(\sigma, \tau; \lambda) = \frac{1}{2} \sigma^2 + \lambda \sigma + \frac{1}{2\beta} \tau^2.
\]

For any given \( u \in U_a \), the virtual work principle gives
the equilibrium equation and boundary conditions for
this stress vector:

\[
\Lambda_t(w)s = \begin{cases} \sigma_x + \tau_y = 0, & (x, y) \in \Omega \\ \int_h^h (\tau + \alpha w_x \sigma) dy = \tilde{f}(x), & x \in [0, L] \\ \tau(x, \pm h) = 0, & x \in [0, L], \\ \int_h^h \sigma(L, y) dy = 0. \end{cases}
\]

(54)

The gap function in this case should be \( G(w, \sigma) = \int_0^L \frac{1}{2} \sigma w_x^2 \) df. Since \( w_x \) can be determined by
integrating (54),

\[
w_x = -\frac{\int_0^x \tilde{f}(t) dt + \int_0^h \tau dy + c}{\alpha \int_0^h \sigma dy},
\]

where \( c = -\int_0^h (\tau + \alpha w_x \sigma)|_{x=0} dy \) is an integral constant, the gap function can be written as

\[
G_a(\sigma, \tau) = \int_0^L \frac{[^{\int_0^x \tilde{f}(t) dt + \int_0^h \tau dy}]^2}{2\alpha \int_0^h \sigma dy}.
\]

(55)

Its sign will depend on the sign of \( \int_h^h \sigma dy \). Since
\( \sigma = v_x + \frac{1}{2} \alpha w_x^2 - \lambda \), if the buckling load \( \lambda \) is big
enough, \( \sigma \) will be negative everywhere in \( \Omega \). For the
clamped/simply supported beam, the statically admissible space is:

\[
S_a = \left\{ \left( \begin{array}{c} \sigma \\ \tau \end{array} \right) \mid \sigma_x + \tau_y = 0, \ \tau(x, \pm h) = 0, \right\}.
\]

(56)

On \( S_a \) the total complementary energy \( P^* \) is then

\[
P^*(\sigma, \tau; \lambda) = -\int_\Omega \left[ \frac{1}{2} \sigma^2 + \lambda \sigma + \frac{1}{2\beta} \tau^2 \right] d\Omega - G_a(\sigma, \tau),
\]

which depends on the stress vector only. The general/ed
complementary energy for this problem is

\[
L(u, s; \lambda) = \int_\Omega \left[ \frac{1}{2} \sigma w_x^2 + \tau(w_x + v_y) \right] d\Omega - \int_0^L \tilde{f} w dx.
\]

(57)

By the triality theorem, the stability criterion for the
post-buckling problem can be proposed as following:

**Theorem 8** For the given load \( \tilde{f}(x) \) and \( \lambda > 0 \), any
critical point \((u, \tilde{s})\) of \( L \) solves the post-buckling problem 2.
If \( G(\tilde{w}, \tilde{\sigma}) \geq 0 \), then \((u, \tilde{s})\) is a stable buckling state.
\( \tilde{u} = (\tilde{v}, \tilde{w}) \) is a global minimizer of \( P \) and \( \tilde{s} = (\tilde{\sigma}, \tilde{\tau}) \) is
a global maximizer of \( P^* \). If \( G(\tilde{w}, \tilde{\sigma}) \leq 0 \), then \((u, \tilde{s})\) is a \( \delta^+\)-critical point of \( L \). In this case, if either \( \tilde{u} \) or \( \tilde{s} \) is a stable
buckling state. However, if either \( \tilde{u} \) or \( \tilde{s} \) is a local
maximizer of \( P \) or \( P^* \), then \((u, \tilde{s})\) is another
stable buckling state.

If we let \( \alpha = 1, e = \Lambda w = \frac{1}{2} w_x^2, \sigma = \sigma - \lambda \), the total
complementary energy dual to (56) should be

\[
P^*(\sigma; \lambda) = -\int_0^L \frac{1}{2} \sigma^2 + \lambda \sigma + \frac{g^2(x)}{2\sigma} dx,
\]

(58)

where \( g(x) = \int_0^x \tilde{f}(t) dt + c, \ c = -w_x \sigma|_{x=0} \). Its
stationary condition gives

\[
\sigma^2 + \lambda = \frac{1}{2} g^2(x).
\]

(59)

For any given load, this algebraic equation has at least
one real solution \( \sigma(x) \), which gives a general analytic
solution for nonlinear equilibrium problem of Ericksen's
bar:

\[
w(x) = -\int_0^x \frac{g(t)}{\sigma(t)} dt + d.
\]

(60)

The integral constants \( c, d \) depend on boundary conditions.
It is easy to prove that there exists a \( \lambda_0 > 0 \) such that for any given \( \lambda > \lambda_0 \), the problem may have
three solutions \((u_1, \sigma_1), (u_2, \sigma_2), (u_3, \sigma_3)\) (see Fig. 1).
If \( \sigma_1 > 0 > \sigma_2 > \sigma_3 \), then by Theorem 8, \( u_1 \) should be
a global minimizer; \( u_2 \) is a local minimizer and \( u_3 \) is
a local maximizer. Detailed discussion of this analytic
solution is given elsewhere. Fig. 2 shows the graphs of $W^*(\sigma) = -\left(\frac{1}{2}\sigma^2 + \lambda\sigma + g(3\sigma)\right)$ (solid line) and its dual function, i.e. the well-known van der Waals energy: $W(w) = \frac{1}{2}(w^2 - \lambda)^2 - gw$ (dashed line). The Lagrangian $L(w, \sigma) = \frac{1}{2}w^2\sigma - (\frac{1}{2}\sigma^2 + \lambda\sigma) - gw$ is shown in Fig. 3.

![Image](https://example.com/image1.png)

**Fig. 1.** Graphs of $h(\sigma) = \sigma^2(\sigma + \lambda)$ and $\frac{1}{2}g^2$

![Image](https://example.com/image2.png)

**Fig. 2.** Graphs of $W^*(\sigma)$ (solid) and $W(w)$ (dashed)

![Image](https://example.com/image3.png)

**Fig. 3.** Lagrangian $L(w, \sigma)$

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**References**


