Canonical dual finite element method for solving post-buckling problems of a large deformation elastic beam

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A B S T R A C T
This paper presents a canonical dual mixed finite element method for the post-buckling analysis of planar beams with large elastic deformations. The mathematical beam model employed in the present work was introduced by Gao in 1996, and is governed by a fourth-order non-linear differential equation. The total potential energy associated with this model is a non-convex functional and can be used to study both the pre- and the post-buckling responses of the beams. Using the so-called canonical duality theory, this non-convex primal variational problem is transformed into a dual problem. In a proper feasible space, the dual variational problem corresponds to a globally concave maximization problem. A mixed finite element method involving both the transverse displacement field and the stress field as approximate element functions is derived from the dual variational problem and used to compute global optimal solutions. Numerical applications are illustrated by several problems with different boundary conditions.

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1. Large deformation beam model and motivations

We are interested in the development of a mixed finite element method for solving large deformation elastic beam problems governed by the following differential equation [4]:

\[ EIw_{xxx} - 2EIw_{xx} w'' + EIw_{xx} - f(x) = 0, \quad \forall x \in (0, L) \]  

(1)

where \( w \) is the transverse displacement field of the beam axis, \( \alpha = 3h(1-v^2) > 0 \) is a positive constant, with \( h \) the beam height and \( v \) the Poisson’s coefficient, \( E \) is the Young’s modulus, \( I = 2h^3/3 \), and \( \lambda = (1+v)(1-v^2)p/E \), with \( p \) the compressive axial force applied on \( x = L \) and \( f(x) = (1-v^2)q(x) \), with \( q(x) \) a vertical distributed load applied over the beam and \( I \) represents the length of the beam in its undeformed configuration, see Fig. 1. The non-linear beam model (1) was first proposed in 1996 [4] and is known in literature as the Gao beam model [1,16,19,28]. This model relies on the following assumptions: (i) the Euler–Bernoulli hypothesis holds, that is, plane sections perpendicular to the beam axis before deformation remain so after deformation and shear deformations are ignored; (ii) the cross sections are initially uniform along the beam and have an axis of symmetry about which bending occurs; and (iii) the beam is under moderately large elastic deformations. Furthermore, in contrast to the Euler–Bernoulli beam model, the present model does not neglect neither the stresses nor the deformations of the cross sections in the lateral direction, which, in fact, are proportional to \( w_x^2 \) and, therefore, have the same order of magnitude as their corresponding axial stresses and deformations. The axial displacement field of the present beam model is shown to obey the following relation [4]:

\[ u_x = - \frac{1}{2} \left( 1 + v \right) w_x^2 \frac{\lambda}{2h(1+v)} \]  

(2)

The total potential energy associated with this problem is the functional \( \Pi_p : \mathcal{U}_L \rightarrow \mathbb{R} \) given by

\[ \Pi_p(w) = \int_0^L \left( \frac{1}{2} EIw_{xx} + \frac{1}{12} Ew_{xx}^2 - \frac{1}{2} EIw_x^2 \right) dx - \int_0^L fw \, dx \]  

(3)

with \( \mathcal{U}_L \) the so-called kinematically admissible space defined by

\[ \mathcal{U}_L = \{ w \in \mathcal{V}^2_a(0,L) | w(0) = w(L) = 0 \} \]  

(4)

The primal variational problem can be stated as: for the given external loads \( f(x) \) and \( \lambda \), find \( w \in \mathcal{U}_L \) such that

\[ (P) : \quad \Pi_p(w) = \inf \{ \Pi_p(w) | w \in \mathcal{U}_L \} \]  

(5)

It is easy to prove that the stationarity condition \( \delta \Pi_p = 0 \) leads to the governing equation (1) as well as to a proper set of natural boundary conditions.

If the beam is in a pre-buckling state, i.e., before the axial load \( p \) reaches the Euler buckling load \( p_c \) given by the eigenvalue...
The total potential energy $\Pi_p(w)$ is convex in $U_0$ and the non-linear differential equation (1) has only one solution. However, if $p > p_c$, the beam is in a post-buckled state. In this case, the total potential energy $\Pi_p(w)$ is non-convex and Eq. (1) may have at most three (strong) solutions at each material point $x \in (0, L)$: two local minimizers, corresponding to the two possible stable buckled states, and one local maximizer, corresponding to an unstable buckled state. These solutions are very sensitive to both the axial load $\lambda$ and the distributed lateral force field $f(x)$. The numerical discretization of a non-convex variational problem leads to a global optimization problem in a finite dimensional space. Due to the lack of a mathematical theory capable of identifying the global minimizer at each numerical iteration, traditional Newton-type direct numerical methods, including the well-known updated Lagrange methods in each numerical iteration, traditional Newton-type direct numerical methods. Also based on this theory, a mixed finite element method has been developed for the analysis of two-dimensional Landau–Ginzburg phase transition problems [13].

The goal of the present paper is to demonstrate the suitability of the canonical duality theory to solve a non-convex beam problem by means of a mixed finite element method. This method involves both the transverse displacement and the stress fields as approximate element functions. As it will be shown, when used in conjunction with the proposed mixed finite element method, the triality theory provides the necessary mathematical tools to compute and classify the critical points of the present non-convex beam problem, in particular to compute their corresponding global optimal solutions.

2. Canonical dual variational problem

By introducing the canonical stress measure $\sigma = E((\pi/3)w^2_x - \lambda)$ and the function spaces

\[ U_0 = \{w \in W^{2,2}(0, L) | w(0) = w(L) = 0, \ or \ w(0) = w_x(0) = 0, \ or \ w(0) = w_x(0) = w(L) = 0 \} \tag{7a} \]

\[ S_a = C^2(0, L) \tag{7b} \]

the Gao–Strang total complementary energy $\Xi : U_0 \times S_a \rightarrow \mathbb{R}$ can be defined as

\[ \Xi(w, \sigma) = \int_0^L \left( \frac{1}{2} E I w^{2}_x + \frac{1}{2} \sigma w_x^2 - \frac{3}{4 E} (\sigma + E \lambda)^2 - f w \right) dx \tag{8} \]

It can be easily shown that the stationarity condition $\delta \Xi = 0$ leads to the following Euler–Lagrange equations:

\[ E I w_{xxx} - (\sigma w_x)_x - f = 0, \quad \forall x \in (0, L) \tag{9a} \]

\[ \frac{1}{2} w_x^2 - \frac{3}{2 E} (\sigma + E \lambda) = 0, \quad \forall x \in (0, L) \tag{9b} \]

It is worth noting that, the first of these equations renders the von Kármán model if the axial stress $\sigma$ satisfies the homogeneous equilibrium differential equation defined by $\sigma_x = 0$, see [4,5].

For a given $\sigma$, the displacement field $w$ can be determined from the linear equation (9a). Hence, a dual energy functional can be obtained by means of the so-called canonical dual transformation as follows:

\[ \Pi_d(\sigma) = (\Xi(w, \sigma))|_{\delta_\sigma \Xi} = 0 \tag{10} \]

\[ \Pi_d(\sigma) = \max_{\delta_\sigma \Xi} \Xi(w, \sigma) = 0 \tag{11} \]

with $\delta_\sigma \Xi$ the Gateaux derivative of $\Xi$ in the direction of $w$. Therefore, the canonical dual variational problem is stated as follows:

\[ \Pi_d(\sigma) = \max_{\Pi_d(\sigma)} | \delta_\sigma \Xi | \sigma \in U_0 \]

with $U_0$ the dual feasible space.
The generalized Gao–Strang complementary gap function for this beam model can be written as
\[ G(w, \sigma) = \int_0^L \left( \frac{1}{2} E w_n^2 + \frac{1}{2} \sigma w_s^2 \right) dx \] (12)

Let us introduce the following space
\[ \mathcal{U}_k^+ = \{ \sigma \in \mathcal{U}_k | G(w, \sigma) \geq 0 \ \forall w \in \mathcal{U}_k \}. \]

By the general theory proposed in [11], we have the following result:

**Theorem 1** (Complementary-dual principle). Problems \((P)\) and \((P^d)\) are canonically dual to each other in the sense that if \( (w, \sigma) \in \mathcal{U}_k \times \mathcal{S}_k \) is a critical point of \( \Xi(w, \sigma) \), the \( w \in \mathcal{U}_k \) is a critical point of \( \Pi_P(w) \), \( \sigma \in \mathcal{U}_k \) is a critical point of \( \Pi_d(\sigma) \), and
\[ \Pi_P(\mathcal{W}) = \Xi(\mathcal{W}, \mathcal{S}) = \Pi_d(\mathcal{S}) \] (13)

Moreover, if \( \sigma \in \mathcal{U}_k^+ \), then \( w \in \mathcal{U}_k \) is a global minimizer of the primal problem \((P)\) and \( \sigma \in \mathcal{U}_k \) is a global maximizer of the canonical dual problem \((P^d)\), i.e.
\[ \Pi_P(\mathcal{W}) = \min_{w \in \mathcal{U}_k} \Pi_P(w) = \max_{\sigma \in \mathcal{U}_k^+} \Pi_d(\sigma) = \Pi_d(\mathcal{S}) \] (14)

This theorem shows that the total complementary energy \( \Xi(w, \sigma) \) is a saddle-point functional on \( \mathcal{U}_k \times \mathcal{U}_k^+ \). Based on this theorem, a mixed finite element method can be developed to find the global minimizer of the non-convex primal problem \((P)\).

### 3. General canonical dual finite element method

Let the domain \( \Omega = (0, l) \) be discretized into finite elements \( \Omega^e \) such that \( \Omega = \bigcup_{e=1}^m \Omega^e \) with \( \Omega^e = (x_e, x_{e+1}) \) and \( m \) the number of elements. Further, let \( (w_h^e(x), \sigma_h^e(x)) \in \mathcal{U}_k \times \mathcal{S}_k \) be a pair of approximating element functions defined as
\[ w_h^e(x) = N_h^e w_c \] (15a)
\[ \sigma_h^e(x) = N_h^e s_c \] (15b)
with \( \mathcal{U}_k \) and \( \mathcal{S}_k \) finite-dimensional subspaces of \( \mathcal{U}_k \) and \( \mathcal{S}_k \), respectively. \( N_h \) and \( s_c \) are vectors collecting suitable displacement and stress approximate functions, respectively, whereas \( w_c \in \mathcal{W}_k \) and \( s_c \in \mathcal{S}_k \) are the vectors of element unknown displacement and stress parameters, respectively, with \( n_d \) and \( n_s \) the number of displacement and stress element parameters, respectively.

The discretized form of the Gao–Strang total complementary energy can be written as
\[ \Xi_h(w_c, s_c) = \sum_{e=1}^m \left( \frac{1}{2} w_h^e G_c(s_c) w_h^e - \frac{1}{2} s_h^e K_c s_h^e - \lambda^e_{c} s_h^e - f^e_{c} s_h^e - c^e \right) \] (16)
where \( G_c(s_c) \in \mathcal{R}^{n_s} \times \mathcal{R}^{n_s} \) is the Hessian matrix associated with the Gao–Strang gap function, which is defined by
\[ G_c(s_c) = \int_{\Omega^e} \left( E \mathcal{N}_h^e N_h^e + (N_h^e s_c) N_h^e s_c \right) dx \] (17)
\[ K_c \in \mathcal{R}^{n_s} \times \mathcal{R}^{n_s} \] is a positive-definite matrix defined by
\[ K_c = \int_{\Omega^e} \frac{3}{2} E \mathcal{N}_h^e \mathcal{N}_h^e dx \] (18)
and \( \lambda_c \in \mathcal{R}^{n_s} \), \( f_c \in \mathcal{R}^{n_s} \) and \( c_c \in \mathcal{R} \) are defined by
\[ \lambda_c = \int_{\Omega^e} \frac{3}{2} \mathcal{N}_h^e(x) dx \] (19a)
\[ f_c = \int_{\Omega^e} f_h N_h dx \] (19b)
\[ c_c = \int_{\Omega^e} \frac{3}{4\pi} \lambda E dx \] (19c)

After assembling, the discretized form of the Gao–Strang total complementary energy can be rewritten as
\[ \Xi_h(w, s) = \frac{1}{2} w G(s) w - \frac{1}{2} s K s - \lambda s - f s - c \] (20)
where \( G(s) \in \mathcal{R}^{n_s} \times \mathcal{R}^{n_s} \), \( K \in \mathcal{R}^{n_s} \times \mathcal{R}^{n_s} \), \( w \in \mathcal{R}^{n_s} \), \( s \in \mathcal{R}^{n_s} \), \( \lambda \in \mathcal{R}^{n_s} \), \( f \in \mathcal{R}^{n_s} \) and \( c \in \mathcal{R} \) are now assembled, or global, forms of their corresponding element entities introduced above, with \( n \) and \( m \) the number of total displacement and stress unknown parameters, respectively.

The stationarity condition \( \delta \Xi_h(w, s) = 0 \) leads to the canonical Euler–Lagrange equations given by
\[ G(s) w - f = 0 \] (21a)
\[ K s - \lambda = 0 \] (21b)

The finite-dimensional space \( \mathcal{U}_k^+ \) is defined by
\[ \mathcal{U}_k^+ = \{ s \in \mathcal{R}^{n_s} | G(s) \text{ is invertible} \} \] (22)
If \( G(s) \) is invertible, Eq. (21a) can be rewritten as
\[ w = G^{-1}(s)f \] (23)
On insertion of this equation into (20) leads to the discretized canonical dual function \( \Pi_d^h(w) \) defined by
\[ \Pi_d^h(s) = -\frac{1}{2} s^T G^{-1}(s) f - \frac{1}{2} s^T K s - \lambda s - f s - c \] (24)

On the other hand, the discretized total potential energy \( \Pi_P^h(w) \), well defined on the discretized kinematically admissible space \( \mathcal{U}_k^h \), can be obtained by
\[ \Pi_P^h(w) = \max_{\mathcal{U}_k^+} \Xi_h(w, s) \] (25)

Since both \( \Pi_P^h(w) \) and \( \Pi_d^h(s) \) are non-convex, they might have multiple critical points. Using the triality theorem proposed by Gao [5,7], the global maximizer of \( \Pi_P^h(w) \) can be determined by restricting \( \mathcal{U}_k^h \) to its subspace:
\[ \mathcal{U}_k^+ = \{ s \in \mathcal{R}^{n_s} | G(s) \text{ is positive-definite} \} \] (26)
The other local extrema can be determined by restricting \( \mathcal{U}_k^+ \) to its subspace given by
\[ \mathcal{U}_k^- = \{ s \in \mathcal{R}^{n_s} | G(s) \text{ is negative-definite} \} \] (27)

By the triality theorem proposed in the general framework of global optimization [12], we have the following result:

**Theorem 2.** If \( s \) is a critical point of the discretized canonical dual function \( \Pi_d^h(s) \), then
\[ w = G^{-1}(s)f \] (28)
is a critical point of \( \Pi_P^h \) and
\[ \Pi_P^h(w) = \Pi_d^h(s) \] (29)
The critical solution \( w \) is a global minimizer of \( \Pi_P^h(w) \) over \( \mathcal{U}_k^+ \) if and only if \( s \) is a global maximizer of \( \Pi_d^h(s) \) over \( \mathcal{U}_k^+ \), i.e.,
\[ \Pi_P^h(w) = \min_{\mathcal{U}_k^+} \Pi_P^h(w) \iff \Pi_d^h(s) = \max_{\mathcal{U}_k^+} \Pi_d^h(s) \] (30)
If \( s \in \mathcal{U}_k^+ \), then \( w \) is a local maximizer of \( \Pi_P^h \) if and only if \( s \) is a local maximizer of \( \Pi_d^h(s) \), i.e., over a neighborhood \( N_h \subset \mathcal{U}_k^+ \) we have
\[ \Pi_P^h(w) = \max_{h \in N_h} \Pi_P^h(w) \iff \Pi_d^h(s) = \max_{s \in N_s} \Pi_d^h(s) \] (31)
If \( s \in \mathcal{U}_k^- \) and \( n_d = n_s \), then \( w \) is a local minimizer of \( \Pi_P^h \) if and only if \( s \) is a local minimizer of \( \Pi_d^h(s) \), i.e., over a neighborhood
$\lambda_{k}^{h} \subset u_{k}^{h}$ and $\lambda_{s}^{h} \subset u_{s}^{h}$ we have
\[ P_{\lambda}^h(\mathbf{w}) = \min_{\mathbf{w} \in \lambda_{k}^{h}} P_{\lambda}^h(\mathbf{w}) = \min_{\mathbf{w} \in \lambda_{s}^{h}} P_{\lambda}^h(\mathbf{w}) \quad (32) \]

The triality theory shows that both the min-max duality and the double-max duality statements, (30) and (31), respectively, hold strongly, whereas the double-min duality statement (32) holds conditionally with $n_{w} = n_{s}$. If $\mathbf{w} \in u_{k}^{h}$ but $n_{w} \neq n_{s}$, then the double-min duality statement holds weakly (see [12]). The canonical dual function $P_{\lambda}^h(\mathbf{s})$ is actually the discretized pure complementary energy proposed by Gao in 1999 [6]. The associated variational principle is known as the Gao principle in finite deformation theory [18]. Based on this principle and the triality theorem, we have the following result.

**Theorem 3.** For any given mixed finite element discretization (15b), the pure complementary energy $P_{\lambda}^h(\mathbf{s})$ has at most one critical point $\mathbf{s} \in u_{k}^{h}$ and the vector $\mathbf{w} = \mathbf{G}^{-1}(\mathbf{s})\mathbf{f}$ is the unique global minimizer of the discretized total potential energy $P_{\lambda}^h(\mathbf{w})$. If $n_{w} = n_{s}$, the pure complementary function $P_{\lambda}^h(\mathbf{s})$ has at most one local minimizer and one local maximizer in $u_{k}^{h}$, the associated vector $\mathbf{w}(\mathbf{s})$ should be also local min and local max of $P_{\lambda}^h(\mathbf{w})$, respectively.

This theorem should play important roles in large-scale finite element analysis for large deformation mechanics.

**3.1. A particular case—polynomial cubic $w_{e}^{h}$ and constant $\sigma_{e}^{h}$**

We present in this section a particular canonical dual finite element method based on piecewise-cubic polynomial transverse displacements and piecewise-constant stresses.

The vectors collecting the approximate element functions for the displacement and stress fields are assumed as
\[
\mathbf{N}_{w} = \begin{bmatrix} \frac{1}{3}(1-\xi)^{3}(2+\xi) \\ \frac{1}{3}(1-\xi)^{3}(1+\xi) \\ \frac{1}{3}(1+\xi)^{3}(2-\xi) \\ \frac{1}{3}(1+\xi)^{3}(1-\xi) \end{bmatrix}, \quad \mathbf{N}_{s} = [1] \quad (33)
\]

with $\xi = 2x/L_{e} - 1$, where $L_{e}$ represents the length of the beam element $e$ in its undeformed configuration. Note that the displacement element functions are exactly the same as the ones employed within the standard primal (displacement-based) finite element model typically used for Euler–Bernoulli beam problems.

Using these approximations leads to the following element matrices and vectors
\[
\mathbf{G}_{e}(\sigma_{e}^{h}) = \begin{bmatrix} \frac{12C_{L}}{L_{e}^{2}} + 6\sigma_{e}^{h} & 6\sigma_{e}^{h} & \frac{6C_{L}}{L_{e}} + \sigma_{e}^{h} & -\frac{12C_{L}}{L_{e}^{2}} - 6\sigma_{e}^{h} & 6\sigma_{e}^{h} + \sigma_{e}^{h} \\ 6\sigma_{e}^{h} & \frac{12C_{L}}{L_{e}^{2}} + 6\sigma_{e}^{h} & \frac{12C_{L}}{L_{e}} + \sigma_{e}^{h} & 6\sigma_{e}^{h} & \frac{12C_{L}}{L_{e}^{2}} + \sigma_{e}^{h} \\ \frac{6C_{L}}{L_{e}} + \sigma_{e}^{h} & \frac{12C_{L}}{L_{e}^{2}} + 6\sigma_{e}^{h} & \frac{12C_{L}}{L_{e}} + \sigma_{e}^{h} & \frac{12C_{L}}{L_{e}^{2}} + \sigma_{e}^{h} & \frac{6C_{L}}{L_{e}} + \sigma_{e}^{h} \\ -\frac{12C_{L}}{L_{e}^{2}} - 6\sigma_{e}^{h} & 6\sigma_{e}^{h} & \frac{12C_{L}}{L_{e}} + \sigma_{e}^{h} & \frac{12C_{L}}{L_{e}^{2}} + \sigma_{e}^{h} & \frac{6C_{L}}{L_{e}} + \sigma_{e}^{h} \\ 6\sigma_{e}^{h} & \frac{6C_{L}}{L_{e}} + \sigma_{e}^{h} & \frac{12C_{L}}{L_{e}} + \sigma_{e}^{h} & 6\sigma_{e}^{h} & \frac{12C_{L}}{L_{e}^{2}} + \sigma_{e}^{h} \end{bmatrix} \quad (34)
\]
\[
\mathbf{K}_{e} = \frac{3L_{e}}{2EI} \quad (35)
\]
\[
\lambda_{e} = \frac{3L_{e}C_{L}}{2EI} \quad (36)
\]
\[
\mathbf{f}_{e} = \begin{bmatrix} \frac{6}{L_{e}^{2}} \\ \frac{6}{L_{e}^{2}} \\ \frac{6}{L_{e}^{2}} \\ \frac{6}{L_{e}^{2}} \end{bmatrix} \quad (37)
\]
\[
c_{e} = \frac{3EI\lambda^{2}}{4x} \quad (38)
\]

It is worth noting that, the components of $\mathbf{G}_{e}$ which do not depend on $\sigma_{e}^{h}$ match exactly those of the stiffness matrix associated with the standard displacement-based Euler–Bernoulli beam element also with cubic polynomial transverse displacements.

It is also worth noting that the necessary condition for the stability of any mixed finite element method, which reads [29]
\[
n_{w} \geq n_{w} - r \quad (39)
\]

is satisfied by the proposed canonical dual finite element method. For a single element, we have, in this case, $n_{e} = 1$, $n_{w} = 4$ and $r = 3$. Thus, it can be easily verified that the condition given above is fulfilled either for a single element or a patch of elements with appropriate boundary conditions.

As a final remark, we note that the present method could be straightforwardly extended to different finite element approximation schemes.

### 4. Numerical applications

We present in this section several beam problems with distinct boundary conditions. The problems were analyzed in three different stages: first, a comparison between the present beam model and the Euler–Bernoulli beam model is considered; second, the three critical points of each problem are identified and computed. Finally, the proposed canonical dual mixed finite element method along with the triality theory are employed in order to compute the global optimal solution of each problem. In this last stage, four different meshes are considered, in particular, meshes with 2, 3, 4 and 5 finite element. The canonical dual problems were solved by resorting to the BFGS method. The following material and geometrical data were kept fixed for all computations: $\nu = 0.3$, $E = 10000$, $h = 0.1$, $L = 1$.

**4.1. Cantilever beam with uniform distributed load**

A cantilever beam problem is analyzed in this section. The beam is clamped at $x = 0$ and free at $x = L$. The critical load of the cantilever beam is given by $\lambda_{c} = 0.0027$.

First, a comparison of the results obtained using the present beam model and the Euler–Bernoulli beam model were carried out. For this comparison, $\lambda$ was kept null and the applied transverse load was set to be $f = 1$. For this load case, the solution of the problem is unique. The deflections obtained using both models are depicted in Fig. 2.
The solutions were computed using the primal (or displacement-based) finite element method based on piecewise-cubic polynomial functions for the displacements. Only a single finite element was considered for the computations.

In the second part of this problem, the cantilever beam was assumed to be under the action of both transverse and axial loads. The load values were taken as $f=0.1$ and $\lambda = 0.003 > \lambda_c$. The three solutions of this problem were computed using also the primal finite element method with piecewise-cubic polynomial functions for the displacements. The obtained approximate deflections of the beam are depicted in Fig. 3.

In the last part of this problem, and taking again $f=0.1$ and $\lambda = 0.003 > \lambda_c$, the proposed canonical dual finite element method was applied for the computation of the global maximizers of $\Pi_0^d$. Meshes with 2, 3, 4 and 5 finite elements were employed. The obtained final deformed configurations of the beam are represented in Fig. 4. As it can be seen, only solutions similar to the one corresponding to the global minimizer of $\Pi_0^d$ were computed. It can also be seen that the solutions obtained with the various meshes are very close. Table 1 indicates the axial displacement, the transverse displacement, the rotation and also the moment at the free end of the cantilever obtained using the various meshes. It is evident how close the solutions are and how well they converge to the exact solution. Finally, Table 2 presents the numerical values of the total potential and dual energy functionals obtained using the canonical dual method. Clearly, while the total potential energies converge from above, the dual energies converge from below to the exact solution. This stems from the fact that the computed solutions correspond to global minimizers of $\Pi_0^d$ and also to global maximizers of $\Pi_0^d$. It can also be seen that, with only five finite elements, the canonical dual method gives an approximate solution with an error in energy which is lower than 0.7%. Further, as expected, the moment at the free end tends to zero along with the refinement of the mesh.

### Table 2
Energies of the cantilever beam with uniform load.

<table>
<thead>
<tr>
<th></th>
<th>2FE</th>
<th>3FE</th>
<th>4FE</th>
<th>5FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pi_0^d$</td>
<td>-0.007110</td>
<td>-0.007130</td>
<td>-0.007134</td>
<td>-0.007135</td>
</tr>
<tr>
<td>$\Pi_0^d$</td>
<td>-0.007450</td>
<td>-0.007272</td>
<td>-0.007212</td>
<td>-0.007185</td>
</tr>
</tbody>
</table>

**Fig. 3.** Critical points of the cantilever beam: global minimizer (in green), local minimizer (in red), local maximizer (in blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Fig. 4.** Deformed configurations of the cantilever beam with uniform load.

![Deformed configurations of the cantilever beam](image)

**Table 1**
Displacements of the cantilever beam with uniform load.

<table>
<thead>
<tr>
<th></th>
<th>2FE</th>
<th>3FE</th>
<th>4FE</th>
<th>5FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(L)$</td>
<td>-0.020303</td>
<td>-0.020037</td>
<td>-0.019952</td>
<td>-0.019912</td>
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<tr>
<td>$w(L)$</td>
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<td>0.104930</td>
<td>0.104411</td>
<td>0.104173</td>
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<td>$w_y(L)$</td>
<td>0.155715</td>
<td>0.152772</td>
<td>0.151861</td>
<td>0.151452</td>
</tr>
<tr>
<td>$w_y(L)$</td>
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<td>-0.000946</td>
<td>-0.000608</td>
<td>-0.000414</td>
</tr>
</tbody>
</table>

**Fig. 5.** Simply supported beam—comparison: solution for the Euler–Bernoulli model (in red), solution for Gao’s model (in blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**4.2. Simply supported beam**

A simply supported beam problem is analyzed in this section. The beam is fixed in both directions at $x=0$ and fixed only in the $y$-direction at $x=L$. The critical load of the beam is given by $\lambda_c = 0.0078$.

First, a comparison of the results obtained using the present beam model and the Euler–Bernoulli beam model was carried out. For this comparison, $\lambda$ was kept null and the applied transverse load was set to be $f=2$. For this load case, the solution of the problem is unique. The deflections obtained using both models are depicted in Fig. 5. The solutions were computed using the primal finite element method with piecewise-cubic polynomial functions. Only a single finite element was considered for the computations.

In the second part of this problem, the cantilever was assumed to be under the action of both transverse and axial loads. The load values were taken as $f=0.1$ and $\lambda = 0.015 > \lambda_c$. The three solutions of this problem were computed also using the primal finite element method with piecewise-cubic polynomial functions. The obtained approximate deflections of the beam are depicted in Fig. 6.

In the last part of this problem, and taking again $f=0.1$ and $\lambda = 0.015 > \lambda_c$, the proposed canonical dual finite element method was applied for the computation of the global maximizers of $\Pi_0^d$. Meshes with 2, 3, 4 and 5 finite elements were employed. The obtained final deformed configurations of the beam are represented in Fig. 7. As it can be seen, only solutions similar to the one corresponding to the global minimizer of $\Pi_0^d$ obtained in the second part of this problem were computed. It can also be seen
First, a comparison of the results obtained using the present beam model and the Euler–Bernoulli beam model were carried out. For this comparison, \( \lambda \) was kept null and the applied transverse load was set to be \( f = 5 \). For this load case, the solution of the problem is unique. The deflections obtained using both models are depicted in Fig. 8. The solutions were computed using the primal finite element method with piecewise-cubic polynomial functions for the displacements. Only a single finite element was considered for the computations.

In the second part of this problem, the cantilever was assumed to be under the action of both transverse and axial loads. The load values were taken as \( f = 0.1 \) and \( \lambda = 0.022 > \lambda_c \). The three solutions of this problem were computed also using the primal finite element method with piecewise-cubic polynomial functions. The obtained approximate deflections of the beam are depicted in Fig. 9.

In the last part of this problem, and taking again \( f = 0.1 \) and \( \lambda = 0.015 > \lambda_c \), the proposed canonical dual finite element method was applied for the computation of the global maximizers of \( P^h \). Meshes with 2, 3, 4 and 5 finite elements were employed. The obtained final deformed configurations of the beam are represented in Fig. 10. As it can be seen, only solutions similar to the one corresponding to the global minimizer of \( P^h \) obtained in the second part of this problem were computed. It can also be seen that the solutions obtained with the various meshes are very close. Table 3 indicates the axial displacement, the rotation and also the moment of the beam at \( x = L \) obtained using the various meshes. Finally, Table 4 presents the numerical values of the total potential and dual energy functionals obtained using the canonical dual method. Clearly, while the total potential energies converge from above, the dual energies converge from below to the exact solution. This shows indeed that the computed solutions are in fact not only global minimizers of \( P^h \) but also global maximizers of \( P^d \). It can also be seen that, with only five finite elements, the canonical dual method gives an approximate solution with an error in energy which is lower than 3.8%.

4.3. Clamped/simply supported beam

A clamped/simply supported beam problem is analyzed in this section. The beam is clamped at \( x = 0 \) and fixed only in the y-direction at \( x = L \). The critical load of the cantilever beam is given by \( \lambda_c = 0.017 \).

First, a comparison of the results obtained using the present beam model and the Euler–Bernoulli beam model were carried out. For this comparison, \( \lambda \) was kept null and the applied transverse load was set to be \( f = 5 \). For this load case, the solution of the problem is unique. The deflections obtained using both models are depicted in Fig. 8. The solutions were computed using the primal finite element method with piecewise-cubic polynomial functions for the displacements. Only a single finite element was considered for the computations.

In the second part of this problem, the cantilever was assumed to be under the action of both transverse and axial loads. The load values were taken as \( f = 0.1 \) and \( \lambda = 0.022 > \lambda_c \). The three solutions of this problem were computed also using the primal finite element method with piecewise-cubic polynomial functions. The obtained approximate deflections of the beam are depicted in Fig. 9.

In the last part of this problem, and taking again \( f = 0.1 \) and \( \lambda = 0.015 > \lambda_c \), the proposed canonical dual finite element method was applied for the computation of the global maximizers of \( P^h \). Meshes with 2, 3, 4 and 5 finite elements were employed. The obtained final deformed configurations of the beam are represented in Fig. 10. As it can be seen, only solutions similar to the one corresponding to the global minimizer of \( P^h \) obtained in the second part of this problem were computed. It can also be seen that the solutions obtained with the various meshes are very close. Table 3 indicates the axial displacement, the rotation and also the moment of the beam at \( x = L \) obtained using the various meshes. Finally, Table 4 presents the numerical values of the total potential and dual energy functionals obtained using the canonical dual method. Clearly, while the total potential energies converge from above, the dual energies converge from below to the exact solution. This shows indeed that the computed solutions are in fact not only global minimizers of \( P^h \) but also global maximizers of \( P^d \). It can also be seen that, with only five finite elements, the canonical dual method gives an approximate solution with an error in energy which is lower than 3.8%.

4.3. Clamped/simply supported beam

A clamped/simply supported beam problem is analyzed in this section. The beam is clamped at \( x = 0 \) and fixed only in the y-direction at \( x = L \). The critical load of the cantilever beam is given by \( \lambda_c = 0.017 \).
close. Table 5 indicates the axial displacement, the rotation and also the moment of the beam at \( x=L \), obtained using the various meshes. Finally, Table 6 presents the numerical values of the total potential and dual energy functionals obtained using the canonical dual method for the various meshes. Clearly, while the total potential energies converge from above, the dual energies converge from below to the exact solution. This shows indeed that the computed solutions are in fact not only global minimizers of \( I_P^b \) but also global maximizers of \( I_P^b \). It can also be seen that, with only five finite elements, the canonical dual method gives an approximate solution with an error in energy which is lower than 7.4%.

### 4.4. Cantilever beam with non-uniform distributed load

A cantilever beam with a non-uniform distributed load is analyzed in this section. The beam is clamped at \( x=0 \) and free at \( x=L \). The axial load was taken as \( \lambda = 0.01 > \lambda_c = 0.0027 \), whereas the distributed load was assumed as a symmetric non-uniform function defined by \( f(x) = -200x^2 + 200x - 20 \). As it will be shown, the deformed configuration of the cantilever beam with this non-uniform loading has a more complex shape than the one resulting from a uniform \( f \), see Section 4.1.

The canonical dual finite element method was applied for the computation of the global maximizers of \( I_P^b \). Meshes with 2, 3, 4 and 5 finite elements were employed. The obtained final deformed configurations of the beam are represented in Fig. 11. Table 7 indicates the axial displacement, the transverse displacement, the rotation and also the moment at the free end of the cantilever obtained using the various meshes. The numerical values of the total potential and dual energy functionals obtained using the various meshes are indicated in Table 8. Clearly, while the total potential energies converge from above, the dual energies converge from below to the exact solution. This stems from the fact that the computed solutions correspond to global minimizers of \( I_P^b \) and also to global maximizers of \( I_P^b \). It can also be seen that, with only five finite elements, the canonical dual method gives an approximate solution with an error in energy which is lower than 1.8%.

### 5. Conclusions

A canonical dual mixed finite element method has been proposed for large deformation analysis of planar beams. The beams were modeled using the so-called Gao’s beam model. As the primal variational problem associated with this problem is in general non-convex, traditional direct numerical approaches derived from this variational formulation are in general not suitable to determine the global minimum. A general canonical dual mixed finite element method has been introduced and specialized to the case of piecewise-cubic polynomial transverse displacements and piecewise-constant canonical stresses. It has been shown that the proposed canonical dual mixed finite element method in conjunction with the trilality theory can be used to easily compute global minimum solutions. Several numerical problems have been solved and analyzed using the proposed method. Due to its inherent capabilities, we believe that this method may potentially bring new insight into the field of non-convex computational mechanics.

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### References