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What is This?
Shear Control and Analytic Solutions for 2-D Dynamic Smart Beam Theory

DUXING CAI AND DAVID YANG GAO*

Department of Mathematics, Virginia Polytechnic Institute & State University, Blacksburg, VA 24061

ABSTRACT: Analytic solutions are presented for the boundary control of 2-dimensional dynamic smart beam model with shear actuators on the top and bottom of the beam. By using Fourier Series method, the vibration control, optimal location of actuators, and shape control of smart beam are obtained. Applications are also illustrated.

INTRODUCTION

VIBRATION control of smart beam has been studied extensively in recent years (see, for example, Chui and Cheong, 1996; Choi, 1995; Gao and Russell, 1994; Hodges and Bless, 1995; Kwak, Denoyer and Sciulli, 1995; Lee and Lee, 1996; Milford and Asokanathan, 1996; Mucci and Singh, 1995; Panza and Mayne, 1996; Pota and Alberts, 1995; Yang and Lee, 1997). Most of methods in the literature are based on the traditional beam models (such as Euler-Bernoulli beam model and Timoshenko model). Since these beams cannot deal with the external shear loads, only vertical forces are allowed to control the vibration of the beams. For example, Mucci and Singh (1995) applied the method based on traditional model to dampen the vibration caused by vertical disturbance. In recent years, piezoelectric actuators have been widely used in engineering structures. [Yang and Lee (1997) designed a closed-loop neural network to control the vibration of a smart beam with built-in sensors and actuators.] It's known that the attached or embedded actuators will produce shear deformations in the beam. Also, the effect of transverse shear strain cannot be neglected when dealing with deep beams or sandwich beams with low shear modulus, because this effect becomes relatively significant. Although the Timoshenko beam model allows shear deformation, it requires that the plane sections of the beam remain plane after deformation. Recently, Gao (1997) proposed an extended beam model which allows the shear deformation to vary in the lateral direction, i.e., the plane sections of the beam may not remain plane after deformation. Unlike the classical beam models, this new model can handle shear forces on the top and on the bottom of the beam. Based on this new model, the dynamic beam theory subjected to arbitrary vertical loads and horizontal shear forces is established in this paper. By using Fourier Series method, analytic solutions are obtained. With these results, the optimal vibration control and shape control of the beam via shear forces are also discussed.

DYNAMIC BEAM MODEL AND ANALYTIC SOLUTION

We begin our discussion with the following extended beam model.

As shown in Figure 1, the undeformed elastic beam occupies in the x-y plane a rectangular region: \( \Omega = \{(x,y)|0 \leq x \leq L, -h \leq y \leq h\} \). The beam is subjected to a vertical load \( p(x,t) \) and horizontal loads \( u_1(x,t) \) on the top and \( u_2(x,t) \) on the bottom of the beam. The horizontal and lateral displacements of the beam are described by means of two functions \( v(x,y,t) \) and \( w(x,t) \) respectively. For small plane elastic deformation, the strain vector can be written as

\[
\epsilon_x = \epsilon_x, \quad \epsilon_y = w_y = 0, \quad \gamma = v_y + w_x
\]

where \( \epsilon_x = \frac{\partial v}{\partial x}, \epsilon_y = \frac{\partial v}{\partial y}, w_x = \frac{\partial w}{\partial x}, w_y = \frac{\partial w}{\partial y}, \gamma = \frac{\partial^2 v}{\partial x \partial y}, \) etc.

Then, the kinetic energy and the potential energy of the beam are given by \( T(w,v,t) \) and \( P(w,v,t) \):

\[
T(w,v) = \frac{1}{2} \int_{\Omega} \rho(x,y) w_t^2(x,t) d\Omega
+ \frac{1}{2} \int_{\Omega} \rho(x,y) v_t^2(x,y,t) d\Omega
\]

\[
P(w,v) = \frac{1}{2} \int_{\Omega} [E v_y^2(x,y,t) + G(v_y(x,y,t))
+ w_x(x,t))^2] d\Omega - \int_{0}^{T} \{ p(x,t) w(x,t)
+ u_1(x,t) v(x,h,t) + u_2(x,t) v(x,-h,t) \} dx
\]

The Hamiltonian of this system is:

\[
H(w,v) = \int_{0}^{T} [T(w,v) - P(w,v)] dt
\]

We assume that both ends of the beam are simply supported,
so the stationary condition $\delta H(w, v) = 0$ gives the governing equations

$$Ev_{xx} + Gv_{yy} = \rho v_{tt}$$  \hspace{1cm} (1)

$$2hGw_{xx} + G[v_x(x, h, t) - v_x(x, h, t)] + p = \bar{p}w_{tt}$$  \hspace{1cm} (2)

where $\bar{p}(x) = \int_0^L \rho(x, y)dy$, and the natural boundary conditions

$$G[v_x(x, -h, t) + w_x(x, t)] + u_2(x, t) = 0$$  \hspace{1cm} (3)

$$G[v_x(x, h, t) + w_x(x, t)] - u_1(x, t) = 0$$  \hspace{1cm} (4)

$$v_x(0, y, t) = 0$$  \hspace{1cm} (5)

$$v_x(L, y, t) = 0$$  \hspace{1cm} (6)

The given boundary conditions and initial conditions should be

$$w(0, t) = w(L, t) = 0$$  \hspace{1cm} (7)

$$w(x, 0) = w_x(x, 0) = v(x, y, 0) = v_x(x, y, 0) = 0$$  \hspace{1cm} (8)

This is a second order, coupled partial differential equation system. We are going to solve this system by separating variables. Neglecting the higher order small term $v_x(x, y, t)$ (Gao, 1997), then Equation (1) becomes:

$$Ev_{xx} + Gv_{yy} = 0$$  \hspace{1cm} (9)

Let $v(x, y) = X(x)Y(y)$, from Equations (9), (5), (6) we can get:

$$v(x, y, t) = \sum_{n=0}^{\infty} (C_n e^{\beta_n t} + D_n e^{-\beta_n t}) \cos(\lambda_n x)$$  \hspace{1cm} (10)

with $\beta_n^2 = (E/G)\lambda_n^2, \lambda_n = n\pi/L$. From Equation (2) we have

$$w_{tt} - \alpha^2 w_{xx} = F(x, t), \hspace{1cm} \alpha^2 = \frac{2hG}{\bar{p}}$$  \hspace{1cm} (11)

where, $F(x, t)$ is a nonhomogeneous term depending on $v(x, y, t)$ and $p(x, t)$:

$$F(x, t) = \frac{G}{\bar{p}} [v_x(x, h, t) - v_x(x, -h, t)] + \frac{1}{\bar{p}} p(x, t)$$  \hspace{1cm} (12)

The nonhomogeneous wave Equation (11) with conditions (7) and (8) forms a linear initial valued problem of $w(x, t)$. Let

$$w(x, t) = \sum_{n=1}^{\infty} \omega_n(t) \sin(\lambda_n x),$$  \hspace{1cm} (13)

$$F(x, t) = \sum_{n=0}^{\infty} F_n(t) \sin(\lambda_n x)$$

in which $\omega_n(t)$ and $F_n(t)$ are controlled by

$$\omega_n(t) + \omega_n^2 w_n(t) = F_n(t)$$  \hspace{1cm} (14)

where, $\omega_n = \alpha \lambda_n$. We choose

$$\bar{u}_1(x, t) = \frac{u_1(x, t)}{G} = \sum_{n=0}^{\infty} \bar{u}_{1n}(t) \cos(\lambda_n x)$$  \hspace{1cm} (15)

$$\bar{u}_2(x, t) = -\frac{u_2(x, t)}{G} = \sum_{n=0}^{\infty} \bar{u}_{2n}(t) \cos(\lambda_n x)$$  \hspace{1cm} (16)

and

$$\bar{p}(x, t) = \frac{1}{\bar{p}} p(x, t) = \sum_{n=0}^{\infty} \bar{p}_n(t) \sin(\lambda_n x)$$  \hspace{1cm} (17)

Substituting Equations (10), (13), (15)–(17) into natural conditions (3), (4) and (12) respectively, we get

$$\beta_n C_n e^{\beta_n t} - \beta_n D_n e^{-\beta_n t} + \lambda_n w_n(t) = \bar{u}_{1n}(t)$$  \hspace{1cm} (18)

$$\beta_n C_n e^{-\beta_n t} - \beta_n D_n e^{\beta_n t} + \lambda_n w_n(t) = \bar{u}_{2n}(t)$$  \hspace{1cm} (19)

and

$$F_n(t) = \bar{p}_n(t) = 2a\lambda_n (C_n - D_n) \sinh(\beta_n h)$$  \hspace{1cm} (20)

(here $a = G/\bar{p}$). Then Equation (14) becomes

$$\omega_n^2(t) + \omega_n^2 \omega_n(t) + 2a\lambda_n (C_n - D_n) \sinh(\beta_n h) = \bar{p}_n(t)$$  \hspace{1cm} (21)
From Equations (18), (19), (21), we can determine \( C_n, D_n \) and \( w_n \).

\[
w_n(t) = \frac{1}{k_n} \int_0^t \left[ \bar{p}_n(\tau) - \frac{\alpha_n^2}{\beta_n^2} \tanh(\beta_n h)(\bar{u}_{1n}(\tau) + \bar{u}_{2n}(\tau)) \right] \sin(k_n(t - \tau)) d\tau
\]  
\[
C_n(t) = E_{1n}\bar{u}_{1n}(t) - E_{2n}\bar{u}_{2n}(t) - E_{3n}w_n(t)
\]  
\[
D_n(t) = E_{2n}\bar{u}_{1n}(t) - E_{1n}\bar{u}_{2n}(t) + E_{3n}w_n(t)
\]  

in which,

\[
k_n^2 = \frac{w_n^2}{\beta_n^2} \tan(\beta_n h), \quad E_{1n} = \frac{e^{\beta_n h}}{2\beta_n \sinh(2\beta_n h)}
\]
\[
E_{2n} = \frac{e^{-\beta_n h}}{2\beta_n \sinh(2\beta_n h)}, \quad E_{3n} = \frac{\lambda_n \sinh(\beta_n h)}{\beta_n \sinh(2\beta_n h)}
\]

The final solution of the problem is

\[
w(x,t) = \sum_{n=0}^{\infty} w_n(t) \sin(\lambda_n x)
\]

\[
v(x,y,t) = \sum_{n=0}^{\infty} [C_n(t)e^{\beta_n y} + D_n(t)e^{-\beta_n y}] \cos(\lambda_n x)
\]

For any given external load \( p(x,t) \) and controls \( u_1(x,t) \) and \( u_2(x,t) \), the coefficients \( w_n, C_n \) and \( D_n \) are well defined by Equations (22), (23) and (24) respectively.

**APPLICATION TO CONTROL OF SMART BEAM**

We consider the following three cases.

**Vibration Control of Smart Beam**

The control of vibration of smart beams subjected to arbitrary external loads has been studied by many engineers (Mucci and Singh, 1995; Yang and Lee, 1997). With the results obtained in previous section, we can get the exact solution for the input control to dampen the vibration of beam. To do so, we let \( w_n(t) \equiv 0 \), which yields

\[
\bar{p}_n = \frac{\alpha_n^2}{\beta_n^2} \tanh(\beta_n h)(\bar{u}_{1n} + \bar{u}_{2n}) \equiv 0
\]

\[
\Rightarrow \bar{u}_{1n} + \bar{u}_{2n} = \frac{\beta_n}{2\alpha_n} \coth(\beta_n h)\bar{p}_n
\]

If \( u_{1n} - \bar{u}_{2n} \), i.e., \( u_1(x,t) = -u_2(x,t) \), then the applied controls should be

\[
\bar{u}_{1n} = \bar{u}_{2n} = \frac{\beta_n}{2\alpha_n} \coth(\beta_n h)\bar{p}_n
\]  
\[
u_1(x,t) = -u_2(x,t)
\]

\[
eq \sum_{n=0}^{\infty} \frac{G\beta_n}{2\alpha_n} \coth(\beta_n h)\bar{p}_n(t) \cos(\lambda_n x)
\]

\( \forall x \in [0,L] \)  

From Equation (28) we see that to make \( w(x,t) \equiv 0 \), the accurate applied controls must be continuous functions in domain \([0,L]\). For practical purposes we can put some actuators on the top and on the bottom of the beam respectively to get the approximate controls. Suppose \( 2N \) pieces of actuators are applied on the top and the bottom of the beam at \( x_1, x_2, \ldots, x_N \), respectively, and the corresponding control for the \( i \)th actuator is a \( \delta \)-function \( u_i(x,t) = \delta(x - x_i)\bar{u}_i(t) \), then

\[
u_1(x,t) = \sum_{i=1}^{N} u_i(x,t) = \sum_{i=1}^{N} \delta(x - x_i)\bar{u}_i(t) \]

Let \( u_1(x,t) = \sum_{n=0}^{\infty} G\bar{u}_n(t) \cos(\lambda_n x) \), then

\[
u_1(t) = \frac{2}{GL} \int_0^L u_1(x,t) \cos(\lambda_n x) dx
\]

\[
= \sum_{i=1}^{N} \frac{2}{GL} \bar{u}_i(t) \cos(\lambda_n x_i) \equiv \sum_{i=1}^{N} a_{\lambda_n} \bar{u}_i(t)
\]

where, \( a_{\lambda_n} \equiv (2/GL) \cos(\lambda_n x_i) \). If we let \( u_{1n}(t) = \bar{u}_{1n}(t) \), where \( \bar{u}_{1n}(t) \) depends on \( \bar{p}_n(t), n = 0, \ldots, N - 1 \) are given by Equation (27), i.e.,

\[
A(\bar{x})u_1 = \bar{p}
\]

where, \( \bar{x} = (x_1, x_2, \ldots, x_N) \), and

\[
A(\bar{x}) = \begin{pmatrix}
a_{0,1} & a_{0,2} & \cdots & a_{0,N} \\
a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N-1,1} & a_{N-1,2} & \cdots & a_{N-1,N}
\end{pmatrix}
\]
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(a) without controls

(b) with controls

Figure 2. Vertical displacements w(x,t).

\[
\mathbf{u}_1 = \begin{pmatrix}
\bar{\mathbf{u}}_1^1 \\
\bar{\mathbf{u}}_1^2 \\
\vdots \\
\bar{\mathbf{u}}_1^N \\
\end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix}
\bar{u}_{1,0} \\
\bar{u}_{1,1} \\
\vdots \\
\bar{u}_{1,N-1} \\
\end{pmatrix}
\]

So system (30) will determine the N controls \( u_{1,i} \), \( i = 1, 2, \ldots, N \). Then with such N controls \( u_{1}(x,t), \) \( i = 1, 2, \ldots, N \), the first N terms of the desired accurate controls given by Equation (28) are satisfied. The more accurate control of the smart beam is desired, the more actuators are required. And the controls must satisfy system (30).

Optimal Location of Actuators

To obtain the optimal control of the beam, i.e., with the least cost to get the same result, we must choose the optimal positions \( \bar{x} \) of the applied controls. Using quadratic cost function

\[
C(\mathbf{u}_1) = \sum_{i=1}^{N} (\bar{u}_{1,i})^2
\]

the optimal control problem can be described as follows:

\[
\min \ C(\mathbf{u}_1(\bar{x})) \quad \text{s.t. } A(\bar{x}) \mathbf{u}_1 = \mathbf{p}
\]

Where, \( A = [a_{ij}]_{N \times N}, a_{ij} = (2/\pi) \cos(\lambda_{ij}x_i) \) and \( \bar{x} = (x_1, x_2, \ldots, x_N) \). It’s easy to prove the matrix \( A \) is full rank. And since \( C(\mathbf{u}_1(\bar{x})) \) is convex, problem (32) is well proposed.

Shape Control of Dynamic Smart Beam

Suppose the target shape of the beam is \( \bar{w}(x,t) \), we have

\[
\bar{w}_m(t) = \frac{2}{L} \int_0^L \bar{w}(x,t) \sin(\lambda_n x) dx
\]

From Equation (22), if we let \( \bar{u}_{1,0} = \bar{u}_{2,0} = \bar{u}_m \), the shape control problem is to find control \( u_{1}(x,t) \), such that \( w_m(t) = \bar{w}_m(t) \), i.e.,

\[
\frac{1}{k_n} \int_0^L \left[ \frac{\bar{p}_n - 2a\lambda_n}{\beta_n} \tanh(\beta_n h)(\bar{u}_n) \sin(k_n(t - \tau)) d\tau \right]
\]

\[
= \bar{w}_m(t), \quad n = 0, 1, \ldots
\]

This is an affine system with n unknown \( \bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n) \). For given target function \( \bar{w}(x,t) \), Equation (34) has a unique solution.

**EXAMPLE 1**

Suppose a smart beam is subjected to vertical load \( p(x,t) = x \sin(10 \pi t) \) as Figure 1. At \( x = L/4 \) and \( x = 3L/4 \), four actuators are applied on the top and bottom of the beam respectively to control its vibration. If we choose \( u_1(x,t) = -u_2(x,t) \) and \( \bar{u}_{1,1}(t), \bar{u}_{1,2}(t) \) satisfy Equation (30), then the results are shown as Figures 2–4. We see that, after applying the controls, the maximum vertical displacement of \( w(x,t) \) reduces about 80%, and the maximum horizontal displacement of \( v(x,y,t) \) reduces about 60%. Additionally, if we apply another group of controls, say, as shown in Figure 5, the corresponding results are shown in Figure 6. In this case, the vertical and horizontal displacements are both amplified, not to be dampened any more.

**EXAMPLE 2**

Suppose a smart beam is subjected to vertical load \( p(x,t) = x \sin(10 \pi t) \) as Figure 1. Now we put two actuators on the top and the bottom (the locations \( x_1, x_2 \) are unknown yet) of the beam, respectively, to control its vibration. If choose \( u_1(x,t) = -u_2(x,t) \), and \( \bar{u}_{1,1}(t), \bar{u}_{1,2}(t) \), \( x_1, x_2 \) satisfy Equation (32), then the results are shown as Figures 7–8. From Equation (32), the optimal locations of the actuators are: \( x_1 = 0, x_2 = L \). The corresponding control signals should be given as Figure 7. We see that, if actuators are put at the optimal locations, the amplitudes of the control signals will be less than in Figure 4 by about 30%, yet the damping results are the same; the maxi-
Figure 3. Horizontal displacements $v(x,y,t)$ (at $x = L$, $t = 0.1$).

Figure 4. Controls’ signals.

Figure 5. Second group of controls’ signals.
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Figure 6. $w(x,t)$ and $v(x,y,t)$ (at $x = L, t = 0.1$)

Figure 7. The optimal controls' signals.

Figure 8. $w(x,t)$ and $v(x,y,t)$ (at $x = L, t = 0.1$) under optimal controls in Figure 7.
imum vertical displacement of \( w(x,t) \) reduces to about 80%, and the maximum horizontal displacement of \( v(x,y,t) \) reduces about 60%.

**CONCLUDING REMARKS**

Since the extended beam model can handle shear forces applied on the top and the bottom of the beam, it can be used to study the active control of smart structures with attached/embedded actuators. Figure 3(a) shows that, the horizontal displacement \( v(x,y,t) \) varies in \( y \) even without external shear loads applied on the beam. From examples given in this paper, we can see that if the positions of the applied actuators are given, the best control signals to dampen the vibration of the beam can be exactly determined by Equation (30). On the other hand, if the number of actuators is given, system (32) will give the optimal positions of these actuators and control signals to the optimal control problem. Furthermore, the shape control problem of the smart beam is given by Equation (34), which has a unique solution.

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